JOURNAL OF COMPUTATIONAL PHYSICS **38**, 357–377 (1980)

Numerical Solution of Singular Integral Equations. Application to the Flanged Plane Waveguide*

A. CARON AND M. DUPUY

Laboratoire de Mathématiques Appliquées à la Biomédecine (MEDIMAT), Université Pierre et Marie Curie, 45, rue des Saints-Pères, 75270 Paris Cedex 06, France

AND

CH. PICHOT

Laboratoire des Signaux et Systèmes (CNRS–ESE), Groupe d'Electromagnétisme, Plateau du Moulon, 91190 Gif-sur-Yvette, France

Received May 22, 1979; revised November 28, 1979

Singular integral equations arise frequently in electromagnetic and acoustic theory. We present here a numerical technique for solving integral equations which are meaningful in the sense of Cauchy principal value or Hadamard finite parts by a projection method in Hilbert spaces. Application to the radiation problem from a flanged plane waveguide is given.

I. INTRODUCTION

The integral representation of the diffracted, or radiated, electromagnetic or acoustic field can lead to the introduction of divergent integrals which are meaningful in the sense of the Cauchy principal value or Hadamard finite parts [1, 10].

The numerical evaluation of these integrals is not easy and the methods used by various authors over the last few years have consisted either in isolating the singularity [3, 5, 8] or in some cases in using a regular approximate kernel [6, 7]. The convergence of the results is difficult to reach in the first method: If the interval isolating the singularity is very narrow the results are unstable and the computing time may become prohibitive; on the other hand, if it is very large the approximation in the neighbourhood of the isolation is no longer valid. In the second method the results can be completely wrong.

In this paper, we use a numerical method approximating the distributions of the form: principal value of $1/x$ denoted by v.p. $1/x$ and finite parts of $1/x^m$ denoted p.f. $1/x^m$ with a linear combination of Dirac distributions.

* This research has been supported by the "Centre d'Etudes Théoriques de la Détection et des Communications." Paris, France (Convention No. 77.34.244.00.480.75.01).

0021-9991/80/150357–21$02.00/0

First, we shall examine the problem of radiation from a flanged waveguide. This problem has received considerable attention in recent years with different formalisms or approximations [11, 17]. Then, we shall give the general principles of the numerical method and test it on canonical cases. Finally we shall apply it to the considered problem.

II. Formulation of the Problem

Let us consider a parallel plate flanged waveguide filled with a dielectric with constants ε_1, σ_1, μ_1 radiating into another dielectric medium with constants ε_2, σ_2, μ_2 (Fig. 1a and b). We assume that the field quantities are not y dependent.

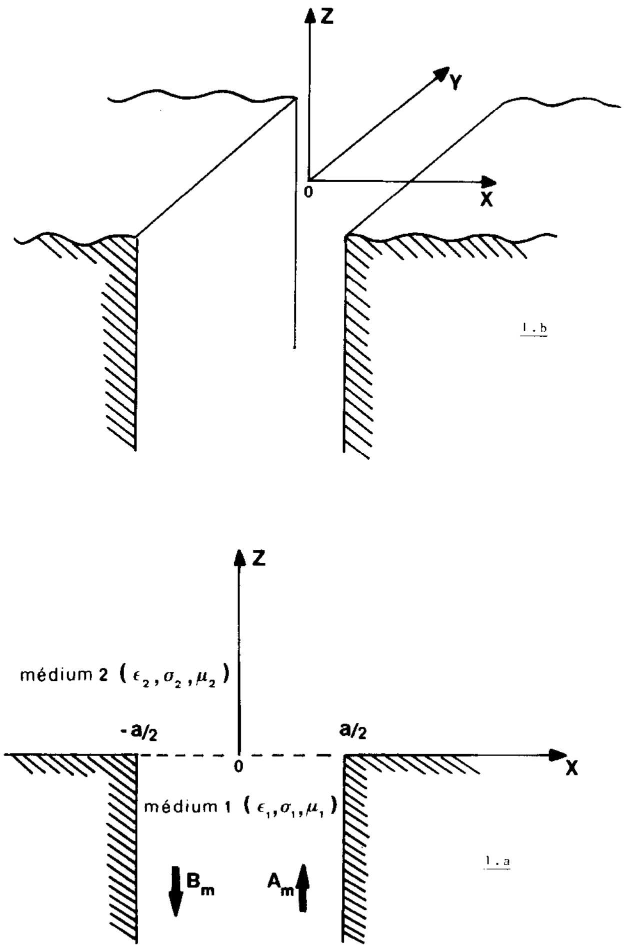


FIG. 1. Geometry of the problem.

All polarization cases are derived from the two elementary polarizations E and H, where the electric and magnetic fields, respectively, are rectilinearly polarized along the y axis.

We shall represent the y component fields by the notation $u(x, z)$. The time dependence $e^{-i\omega t}$ will be omitted.

The integral representation will be established for the E case because it is the only case where divergent integrals appear.

Then we have

$$E_j(x, z) = \{0, u_j(x, z), 0\}, \qquad j = 1, 2,$$
$$H_j(x, z) = \frac{i}{\eta_j k_j} (\hat{e}_y \times \nabla u_j(x, z)), \qquad j = 1, 2, \tag{1}$$

where $\hat{e}_y$ is the unit vector along the y axis. The subscript $j = 1, 2$ denotes the electric magnetic fields in media 1 and 2 (as shown in Figs. 1a and b) and

$$k_j^2 = \varepsilon_j \mu_j \omega^2 + i\sigma_j \mu_j \omega$$
$$= \omega^2 \varepsilon_{c,j} \mu_j,$$

in which (2)

$$\varepsilon_{c,j} = \varepsilon_j + i\sigma_j/\omega,$$
$$\eta_j^2 = \mu_j/\varepsilon_{c,j}.$$

If A_m and B_m are the amplitudes of the incident and reflected modes, respectively, the field may be written in the waveguide as the infinite series of modes

$$u_1(x, z) = \sum_m (A_m e^{-\gamma_m z} + B_m e^{\gamma_m z})\, \psi_m(x).$$

In the expression, the A_m are considered known (emitting situation); the B_m have to be determined.

$$\psi_m(x) = \sin(k_{cm} x + m(\pi/2)), \tag{3}$$

in which

$$\gamma_m = \sqrt{k_{cm}^2 - k_1^2} = \alpha_m - i\beta_m \tag{4}$$

is the propagation constant and $k_{cm} = m\pi/a$ (a is the waveguide width).

For $z > 0$, using the Green's theorem [18], the field may be given by the integral representation

$$u_2(x, z) = 2 \int_{-a/2}^{a/2} u_2(x', 0)\, \partial_{z'} G(x, x'; z, z') \,|_{z'=0}\, dx', \tag{5}$$

where

$$G(x, x'; z, z') = \frac{i}{4} H_0^{(1)}(k_2 \sqrt{(x-x')^2 + (z-z')^2}),$$

in which $H_0^{(1)}$ denotes the Hankel function of first kind and zero order.

When the boundary conditions at $z=0$ are applied, we obtain the infinite system

$$\sum_m L_m(x) B_m = \sum_m K_m(x) A_m, \tag{6}$$

where

$$\begin{aligned} K_m(x) &= \gamma_m \psi_m(x) + F_m(x), \\ L_m(x) &= \gamma_m \psi_m(x) - F_m(x), \end{aligned} \tag{7}$$

in which

$$F_m(x) = \frac{2\mu_1}{\mu_2} \fint_{-a/2}^{a/2} \psi_m(x') \, \partial^2_{zz'} G(x, x'; z, z') \big|_{z=z'=0} \, dx'.$$

The integral in (7) is interpreted in the sense of Hadamard finite parts [1]

$$\left\langle Pf \frac{1}{x^2}, f(x) \right\rangle = \fint_A^B \frac{f(x)}{x^2} \, dx$$

$$= \lim_{\epsilon \to 0} \left\{ \int_A^{-\epsilon} \frac{f(x)}{x^2} dx + \int_\epsilon^B \frac{f(x)}{x^2} dx - \frac{2f(0)}{\varepsilon} \right\}. \tag{8}$$

We note

$$\partial^2_{zz'} G(x, x'; z, z') \big|_{z=z'=0} \simeq \frac{1}{\pi |x-x'|^2} \qquad \text{for} \quad x \to x'. \tag{9}$$

By truncating the series in (6) to M terms and enforcing the resulting equation at the points $x' = x_1, x_2, ..., x_m$ in the interval $[-a/2, a/2]$, we obtain the finite linear system, numerically solved,

$$\sum_{m=1}^{M} L_m(x_p) B_m = \sum_{m=1}^{M} K_m(x_p) A_m, \qquad p = 1, 2, ..., M,$$

from which the B_m can be computed in terms of the known A_m.

III. Principle of the Projection Method in Hilbert Spaces

The main numerical difficulty in the problem described in Section II is the computation of the integrals defined in the sense of Hadamard finite parts.

We make use of the theorem demonstrated in Cherruault [20] and Nissen [21].

p.f. $1/x^m$ belongs to the dual space H^{-m} of the Sobolev space H^m. Then v.p. $1/x$ and v.p. $1/x$ truncated belong to H^{-1}.

Thus, such distributions can be approximated by the method of projection in Hilbert spaces. More precisely, for a given distribution t, we look for an approximate distribution t_N, in the form of a linear combination of Dirac distributions.

$$t_N = \sum_{p=1}^{N} C_p \delta_{a_p}, \tag{10}$$

in which a_p are fixed points of the support of t distribution. t and t_N must coincide on a set of basis function f_q such as

$$f_q = \Lambda^{-1}\delta_{a_q},$$

i.e., the solutions of the equations

$$\Lambda f_q = \delta_{a_q}, \qquad q = 1, 2,..., N, \tag{11}$$

where Λ is the isomorphism between the Sobolev space H^m and its topological dual H^{-m}.

If we denote by E_q the solutions of system (11), we may write

$$\langle t_N, E_q \rangle = \langle t, E_q \rangle, \qquad q = 1, 2,..., N, \tag{12}$$

i.e., the system

$$\sum_{p=1}^{N} C_p E_q(a_p) = \langle t, E_q \rangle, \qquad q = 1, 2,..., N.$$

The calculation of the coefficients C_p related to the distribution p.f. $1/x^2$ and all the convergence demonstrations are described in [20]. But in order to minimize the computation time and memory requirements for the matrix inversion, the relation linking p.f. $1/x^2$ and v.p. $1/x$ applied to a differentiable function ϕ has been used. Thus we have, for p.f. $1/x^2$ truncated at the segment (A, B),

$$\left\langle \text{p.f. } \frac{1}{x^2}, \phi \right\rangle_{[A,B]} = \frac{\phi(A)}{A} - \frac{\phi(B)}{B} + \left\langle \text{v.p. } \frac{1}{x}, \phi' \right\rangle_{[A,B]}. \tag{13}$$

The calculation of p.f. $1/x^2$ is thus reduced to that of v.p. $1/x$ (applied to the derivative of the function) and of the quantities $\phi(A)/A$ and $\phi(B)/B$.

IV. Evaluation of the Coefficients C_p Related to the Distribution v.p. $1/x$

Let Ω be a regular bounded open domain in $\mathbb{R}$. Truncated v.p. $1/x$ belongs to the topological dual of $H^1(\Omega)$ in which

$$H^1(\Omega) = \{u/u \in L^2(\Omega),\ u' \in L^2(\Omega)\},$$

u' being the derivative of u in the distributional sense. The coefficients C_p of approximation

$$\sum_{p=1}^{N} C_p \delta_{a_p}$$

of this distribution are the solutions of the linear system

$$\sum_{p=1}^{N} C_p E_q(a_p) = \left\langle \text{v.p.}\,\frac{1}{x}, E_q \right\rangle, \qquad q = 1, 2, \dots, N,$$

with $E_q \in H^1$ and satisfying $\Lambda E_q = \delta_{a_q}$, in which Λ is now the isomorphism of H^1 on H^{-1}.

If we use the following lemma:

> For every distribution $t \in H^{-m}$, there is one and only one element u of H^m such that
>
> $$\sum_{j=0}^{m} (-1)^j D^{2j} u = t$$

with $D^{2j}u = u^{(2j)}$ in the distributional sense, E_q are solutions of the equation

$$-\frac{d^2 E_q}{dx^2} + E_q = \delta_{a_q}.$$

With the help of the Fourier transform, the solution of this equation is found to be

$$E_q(x) = \tfrac{1}{2} e^{-|x - a_q|}$$

and the system whose solution gives the coefficients C_p related to the distribution v.p. $1/x$ may then be written in the form

$$\sum_{p=1}^{N} C_p e^{-|a_q - a_p|} = \left\langle \text{v.p.}\,\frac{1}{x}, e^{-|x - a_q|} \right\rangle, \qquad q = 1, 2, \dots, N. \tag{14}$$

The right-hand sides of this system are expressed as functions of exponential integrals. In the case of v.p. $1/x$ truncated at $[A, B]$ they may be written in the form

$$s_q = \left\langle \text{v.p.}\,\frac{1}{x}, e^{-|x - a_q|} \right\rangle = \text{v.p.} \int_A^B \frac{e^{-|x - a_q|}}{x}\, dx.$$

Hence

$$s_q = e^{-a_q}[Ei(a_q) - Ei(A)] + e^{a_q}[Ei(-B) - Ei(-a_q)], \tag{15}$$

where $Ei(x) = \int_{-\infty}^{x} (e^t/t)\, dt$ in the sense of principal value if $x > 0$. If points a_q are equidistant and such that

$$a_{q+1} = a_q + h$$

the matrix of the system is Toeplitz and takes the form

$$\begin{pmatrix} 1 & e^{-h} & e^{-2h} & \cdots & e^{-(n-1)h} \\ e^{-h} & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & e^{-2h} \\ \vdots & & \ddots & 1 & e^{-h} \\ e^{-(n-1)h} & \cdots & \cdots & e^{-h} & 1 \end{pmatrix}.$$

Without computing the inverse matrix, we obtain the coefficients C_p related to the distribution v.p. $1/x$ truncated at $[A, B]$ in the form

$$C_p = \frac{(1 + e^{-2h})\, s_p - e^{-h} s_{p-1} - e^{-h} s_{p+1}}{1 - e^{-2h}}, \qquad p \neq 1, n,$$

$$C_1 = \frac{s_1 - s_2 e^{-h}}{1 - e^{-2h}}, \tag{16}$$

$$C_n = \frac{s_n - s_{n-1} e^{-h}}{1 - e^{-2h}}$$

with the s_p obtained by formula (15).

We note that, in order to calculate the coefficient C_{i-1}, we need s_{i-2}, s_{i-1} and s_i. To compute C_i, it is necessary to compute s_{i+1} only, if the values of s_i and s_{i+1} have been stored.

V. Numerical Applications

V.1. *Calculation of* $\langle p.f.\ 1/x^2, \phi \rangle$

In order to show the accuracy of the method and thus to test its validity, the following functions have been used,

(a) $\phi(x) = \cos x$,

(b) $\phi(x) = e^x$,

functions for which tabulations of the exact values of $\langle \text{p.f. } 1/x^2, \phi \rangle$ may be found from sine and exponential integrals.

(c) $\phi(x) = |x|\, Y_1(|x|)$, where Y_1 is the Bessel function of the first kind. $\phi(x)$ is a function whose value may be found by integration of the series representation of Y_1 by Abramovitz and Stegun [19].

The numerical approximation is obtained from

$$\left\langle \text{p.f.}\, \frac{1}{x^2}, \phi \right\rangle_{[A,B]} = \frac{\phi(A)}{A} - \frac{\phi(B)}{B} + \left\langle \text{v.p.}\, \frac{1}{x}, \phi'(x) \right\rangle_{[A,B]}.$$

The coefficient C_p, $p = 1, 2, \ldots, N$, N fixed, related to the distribution v.p. $1/x$ truncated at $[A, B]$ are found from formulas (15) and (16), whence the sum

$$\sum_{p=1}^{N} C_p \phi'(a_p)$$

is computed, and $a_{p+1} = a_p + H$ with $H = (B - A)/(N - 1)$. To compute $\phi'(a_p)$, we have used either

–the explicit expression of ϕ' when it can be obtained easily, or

—a numerical estimation of $\phi'(a_p)$ from the centered difference formula

$$\phi'(a_p) = \frac{\phi(a_p + \varepsilon) - \phi(a_p - \varepsilon)}{2\varepsilon}$$

with $\varepsilon = H$ or $H/10$.

The results are given in Tables I and II.

It is seen that the results found by the methods described here are in excellent agreement with the correct results found by established methods.

V.2. *Application to the Resolution of Singular Integral Equations*

Let us consider a singular integral equation

$$\text{v.p.} \int_{-A}^{A} \frac{\phi(x)}{x - t}\, dx = f(t), \qquad \text{equation of the first kind,}$$

$$\phi(t) + \text{v.p.} \int_{-A}^{A} \frac{\phi(x)}{x - t}\, dx = f(t), \qquad \text{equation of the second kind,}$$

where $f(t)$ is the known function, and ϕ the unknown.

We approximate the Cauchy principal value v.p. $1/(x - t)$ by the projection method in Hilbert spaces.

$$\left\langle \text{v.p.}\, \frac{1}{x - t}, \phi \right\rangle_{[-A,A]} = \text{v.p.} \int_{-A}^{A} \frac{\phi(x)}{x - t}\, dx \simeq \sum_{p=1}^{N} C_p^t \phi(a_p), \tag{17}$$

where $a_p \in [-A, A]$, C_p^t are known. The unknowns are $\phi(a_p)$, $p = 1, 2, \ldots, N$. Thus we obtain a linear system when the relation (17) is written for N values of t:

$$\sum_{p=1}^{N} C_p^{t_k} \phi(a_p) = f(t_k), \qquad k = 1, 2, \ldots, N, \quad t_k \in [-A, A],$$

TABLE I

Approximation of $\langle \text{p.f. } 1/x^2, \cos x\rangle_{[A,B]} = \int_A^B ((\cos x)/x^2)\,dx$ in Double Precision

Number of points N in $[A, B]$	$\int_{-0.1}^{0.1} ((\cos x)/x^2)\,dx$ (V.E. $= -20.09997$)[a]	$\int_{-0.2}^{0.2} ((\cos x)/x^2)\,dx$ (V.E. $= -10.19978$)	$\int_{-2}^{0.1} ((\cos x)/x^2)\,dx$ (V.E. $= -11.44733$)	$\int_{-1}^{0.1} ((\cos x)/x^2)\,dx$ (V.E. $= -2.972771$)	$\int_{-1}^{2} ((\cos x)/x^2)\,dx$ (V.E. $= -2.883725$)	$\int_{-2}^{2} ((\cos x)/x^2)\,dx$ (V.E. $= -2.794679$)
$N = 200$	−20.09997	−10.19978	−11.44728 5×10^{-5}	−2.972730 4.1×10^{-5}	−2.883602 1.23×10^{-4}	−2.794406 2.73×10^{-4}
$N = 500$	−20.09997	−10.19978	−11.44732 1×10^{-5}	−2.972763 8×10^{-6}	−2.883703 2.2×10^{-5}	−2.794633 4.6×10^{-5}
$N = 1000$	−20.09997	−10.19978	−11.44732 1×10^{-5}	−2.972768 3×10^{-6}	−2.883718 7×10^{-6}	−2.794665 1.4×10^{-5}
$N = 2000$	−20.09997	−10.19978	−11.44732 1×10^{-5}	−2.972769 2×10^{-6}	−2.883721 4×10^{-6}	−2.794673 6×10^{-6}
$N = 4000$	−20.09997	−10.19978	−11.44732 1×10^{-5}	−2.972769 2×10^{-6}	−2.883722 3×10^{-6}	−2.794675 4×10^{-6}

[a] V.E.: Exact value.

TABLE II

Approximation of $\langle \text{p.f. } 1/x^2, e^x \rangle_{[A,B]} = \fint_A^B (e^x/x^2)\,dx$ in Double Precision

Number of points N in $[A,B]$	$\fint_{-0.1}^{0.1} (e^x/x^2)\,dx$ (V.E. = −19.8997)	$\fint_{-0.2}^{0.2} (e^x/x^2)\,dx$ (V.E. = −9.799779)	$\fint_{-2}^{0.1} (e^x/x^2)\,dx$ (V.E. = −12.69329)	$\fint_{-1}^{1} (e^x/x^2)\,dx$ (V.E. = −0.971660)	$\fint_{-1}^{2} (e^x/x^2)\,dx$ (V.E.1.111207)	$\fint_{-2}^{2} (e^x/x^2)\,dx$ (V.E. = 1.240939)
$N = 200$	−19.89997	−9.799779	−12.69329	−0.9716514 9×10^{-6}	1.111256 4.9×10^{-5}	1.241019 8×10^{-5}
$N = 500$	−19.89997	−9.799779	−12.69329	−0.971659 1×10^{-6}	1.111213 6×10^{-6}	1.240948 9×10^{-6}
$N = 1000$	−19.89997	−9.799779	−12.69329	−0.9716600	1.111209 2×10^{-6}	1.240938 1×10^{-6}
$N = 2000$	−19.89997	−9.799779	−12.69328 1×10^{-5}	−0.9716600	1.111207	1.240936 3×10^{-6}
$N = 4000$	−19.89997	−9.799779	−12.69328 1×10^{-5}	−0.9716600	1.111207	1.240936 3×10^{-6}

or

$$\phi(t_k) + \sum_{p=1}^{N} C_p^{t_k}\phi(a_p) = f(t_k), \qquad k = 1, 2, \dots, N.$$

If we take for points t_k, $k = 1, \dots, N$, the points a_p which have been used for approximating v.p. $1/(x - t)$, the matrix of the system is

$$\begin{matrix} 1 + C_1^{a_1} & C_2^{a_1} & C_N^{a_1} \\ C_1^{a_2} & 1 + C_2^{a_2} & C_N^{a_2} \\ \vdots & & \vdots \\ C_1^{a_N} & C_2^{a_N} & 1 + C_N^{a_N} \end{matrix}$$

and we use the projection method in Hilbert spaces N times to determine matrix elements. The location of the points $a_1, \dots, a_N$ is sometimes important. For example, the solution of the equation

$$\phi(t) + \text{v.p.} \int_{-1}^{1} \frac{\phi(x)}{x - t}\, dx = \frac{4}{3} + 2t^2 + (t^2 - t) \log \left| \frac{1 - t}{1 + t} \right|$$

is $\phi(x) = x(x^2 - 1)$.

The numerical solution shows pratically that the accuracy increases:

Firstly, when the number of points used in the distribution approximation is larger (columns 1 to 3 in Table III (theoretical demonstration is given in [20]).

Secondly, when the points a_1 and a_N are close to the ends of the interval $(-A, +A)$ (columns 3 to 6 in Table III)—this numerical finding is not yet explained theoretically.

V.3. *Application to the Problem of a Flanged Waveguide*

The problem is the determination of the reflected modes B_m described in Section II in function of incident modes A_m.

Let $(x' - x) = X$; then the evaluation of $F_m(x)$ may be interpreted as p.f. $1/x^2$ truncated at $((-a/2 - x), (a/2) - x))$ applied to the function

$$\phi_m(X) = \frac{ik_2}{2} |X|\, H_1^{(1)}(k_2 |X|) \sin\left(\frac{m\pi x}{a} + \frac{m\pi x}{k_2 a} + m\frac{\pi}{2}\right).$$

The numerical solution is found in the following steps:

We fix a number of modes M, $M = 8, 12, 16, 32, 64$, for instance, and the same number of points x_i, $i = 1, \dots, M$, in the aperture of the guide.

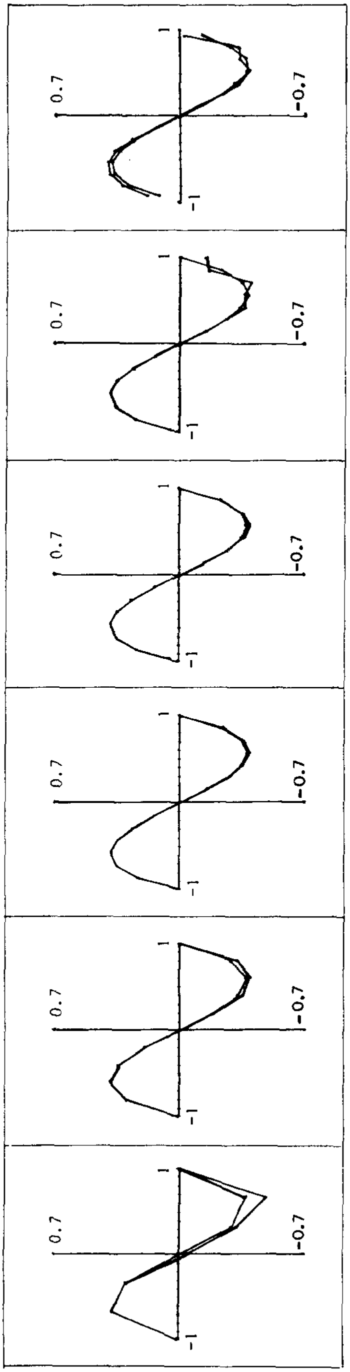
We calculate the elements of the matrix

$$L_m(x_i), \qquad m, i = 1, 2, \dots, M.$$

TABLE III

Solution of the Singular Integral Equation $f(t) + \text{v.p.} \int_{-1}^{1} (f(x)/(x-t))\,dx = -\frac{4}{3} + 2t^2 + (t^3 - t)\log((1-t)/(1+t))$.
The Theoretical Solution is $f(x) = x(x^2 - 1)$

$E = 2 \times 10^{-11}$, $H = 0.3333333$, $N = 7$		$E = 1 \times 10^{-11}$, $H = 0.2$, $N = 11$		$E = 1 \times 10^{-11}$, $H = 0.1428571$, $N = 15$		$E = 7 \times 10^{-4}$, $H = 0.1427552$, $N = 15$		$E = 7 \times 10^{-3}$, $H = 0.1418440$, $N = 15$		$E = 7 \times 10^{-2}$, $H = 0.1333333$, $N = 15$	
Approximate values	Theoretical values	Approximate values	Theoretical values	Approximate values	Theoretical values	Approximate values	Theoretical values	Approximate values	Theoretical values	Approximate values	Theoretical values
+0.0081	+0.0000	+0.0029	+0.0000	+0.0015	+0.0000	+0.0075	+0.0014	+0.0266	+0.0140	+0.1819	+0.1203
+0.3753	+0.3704	+0.2952	+0.2880	+0.2319	+0.2274	+0.2342	+0.2281	+0.2433	+0.2346	+0.3239	+0.2880
+0.2961	+0.2963	+0.3885	+0.3840	+0.3535	+0.3499	+0.3548	+0.3501	+0.3588	+0.3525	+0.3974	+0.3704
−0.0444	0.0	+0.3356	+0.3360	+0.3871	+0.3848	+0.3879	+0.3848	+0.3895	+0.3847	+0.4033	+0.3816
−0.3181	−0.2963	+0.1885	+0.1920	+0.3506	+0.3499	+0.3512	+0.3497	+0.3508	+0.3485	+0.3545	+0.3360
−0.4846	−0.3704	−0.0103	0.0	+0.2614	+0.2624	+0.2620	+0.2622	+0.2627	+0.2609	+0.2627	+0.2477
+0.0013	−0.0000	−0.2023	−0.1920	+0.1376	+0.1399	+0.1380	+0.1398	+0.1370	+0.1390	+0.1451	+0.1310
		−0.3574	−0.3360	−0.0043	0.0	−0.0036	0.0	+0.0004	0.0	+0.0098	0.0
		−0.3955	−0.3840	−0.1452	−0.1399	−0.1446	−0.1398	−0.1469	−0.1390	−0.1184	−0.1310
		−0.3259	−0.2880	−0.2701	−0.2624	−0.2691	−0.2622	−0.2577	−0.2609	−0.2440	−0.2477
		+0.0001	−0.0000	−0.3569	−0.3499	−0.3566	−0.3497	−0.3682	−0.3485	−0.3195	−0.3360
				−0.3965	−0.3848	−0.3950	−0.3848	−0.3660	−0.3847	−0.3898	−0.3816
				−0.3563	−0.3499	−0.3573	−0.3501	−0.4031	−0.3525	−0.3350	−0.3704
				−0.2456	−0.2274	−0.2427	−0.2281	−0.1592	−0.2346	−0.3285	−0.2880
				−0.0000	−0.0000	−0.0040	−0.0014	−0.1459	−0.0140	−0.0209	−0.1203



Characteristics of the approximation: E denotes the distance of the points a_1 and a_N from the ends of $[-1, +1]$. When E decreases, the first and the last points come closer to the bounds. H is the distance between two points. N is the number of points in the interval $[-1, +1]$.

TABLE IV

Resolution of the Flanged Plane Waveguide

N \ B_1	Complex	Modulus	Phase (degrees)
200	−0.8568 −0.0454	0.8580	−177.0
400	−0.8563 −0.0351	0.8570	−177.7
600	−0.8563 −0.0334	0.8570	−177.8
800	−0.8563 −0.0313	0.8569	−177.9
1000	−0.8564 −0.0310	0.8569	−177.9

Notes. Air and water; $\varepsilon_1 = \varepsilon_0$, $\sigma_1 = 0$, $\varepsilon_2 = 70\varepsilon_0$, $\sigma_2 = 15$; frequency $f = 9$GHz; width $a = 2.286$ cm. Values of the amplitude B_1 of the first reflected mode. N = Number of points for approximating the finite parts.

TABLE V

Resolution of the Flanged Plane Waveguide

N \ B_1	Complex	Modulus	Phase (degrees)
200	−0.7291 −0.0650	0.7320	−174.9
400	−0.7292 −0.0560	0.7313	−175.6
600	−0.7292 −0.0545	0.7313	−175.7
800	−0.7294 −0.0526	0.7313	−175.9
1000	−0.7294 −0.0523	0.7313	−175.9

Notes. Dielectric and water; $\varepsilon_1 = 4\varepsilon_0$, $\sigma_1 = 0$, $\varepsilon_2 = 70\varepsilon_0$, $\sigma_2 = 15$; frequency $f = 9$GHz; width $a = 2.286$ cm. Values of the amplitude B_1 of the first reflected mode. N = Number of points for approximating the finite.

TABLE VI

Comparison between the Hilbert and Classical Isolating Singularity Methods

B_1 / M modes	Complex	Modulus	Phase (degrees)	Computer time (IBM 370/168)
8 (Hilbert) $N = 200$	−0.0731 −0.1171	0.1380	58	10 sec
8 (Direct method) $N = 200$	−0.0813 −0.1480	0.1689	61	57 sec
16 (Hilbert) $N = 200$	−0.0721 −0.1174	0.1378	58	25 sec
16 (Direct method) $N = 200$	−0.0732 −0.1448	0.1448	60	1 min 52 sec.

Notes. $\varepsilon_1 = \varepsilon_0$, $\sigma_1 = 0$, $a = 2.286$ cm, $\varepsilon_2 = \varepsilon_0$, $\sigma_2 = 0$, frequency: 8.355 GHz.

Every element involves the calculation of an integral in the Hadamard finite parts sense as well as the right-hand side of the equation.

$$K_1(x_i) = \gamma_1 \phi_1(x_i) + F_1(x_i), \qquad i = 1, 2, \ldots, M,$$

in the particular case $A_1 = 1$ and $A_m = 0$ for $m \neq 1$.

We solve the linear system (M equations – M unknowns) by standard methods.

The results are given in Tables IV and V for two different guides and media: water and air and water and dielectric at a frequency of 9 GHz, taking eight modes into account. Only the first reflected mode which has the most significant value is studied with an increasing number of points.

Note the rapid convergence and stability of the results. The computing time is small: about 15 sec for each 200 supplementary points with an IBM 370/168 computer.

No significant variation of the results has been observed with a larger number of modes. See Figs. 2 and 3.

In Table VI the Hilbert method is compared with the direct method which involves the numerical application of the definition of the finite parts [8]. In this example $\varepsilon = 10^{-3}$ is used with an even Simpson integration algorithm with 200 points. Note the rapid convergence of the first reflected mode B_1 and the small computer time required by Hilbert method ($M = 8$ to 16 modes in each method).

The results are compared to those calculated by other methods of analysis (wedge diffraction [11], Harrington [22] geometrical theory of diffraction (GTD [23])).

The comparisons, shown in Fig. 4, indicate that the results are accurate for a/λ close to 1. As no approximation in the formulation of the physical problem has been considered, the results given by the method of projection in Hilbert spaces may be considered as very accurate.

TABLE VII

Resolution of the Flanged Plane Waveguide

N \ B_1	Complex	Modulus	Phase (degrees)	Computing time	Energy conservation
			8 modes		
200	−0.0731 −0.1171	0.1380	58	10 sec	0.996
1000	−0.0716 −0.1107	0.1318	57	26.5 sec	0.995
2000	−0.0713 −0.1098	0.1310	57	48 sec	0.995
			16 modes		
200	−0.0721 −0.1174	0.1378	58	25 sec	0.999
1000	−0.0708 −0.1116	0.1322	58	1 min 32 sec	0.999
2000	−0.0706 −0.1110	0.1315	58	2 min 56 sec	0.999

Notes. $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$, $\sigma_1 = \sigma_2 = 0$; frequency $f = 8.355$ GHz; width $a = 2.286$ cm. Values of the amplitude B_1 of the first reflected mode. N = Number of points for approximating the finite parts.

A stable convergence of these results is obtained when the number of points in the method increases (see Table VII). The method could be used for any a/λ but a larger a/λ leads, for the same accuracy, to an increasing number of points in the method of projection in Hilbert spaces and consequently computing times become longer.

CONCLUSION

A class of problems involving singular integral equations has been treated in the past by isolating the singularity and evaluating the integral directly according to the definition of Hadamard's finite parts. This method is unstable depending on the parameter range in which the singularity is isolated.

A projection method making use of Hilbert spaces has been developed. It has been applied successfully to the calculations of finite parts, the solution of a singular

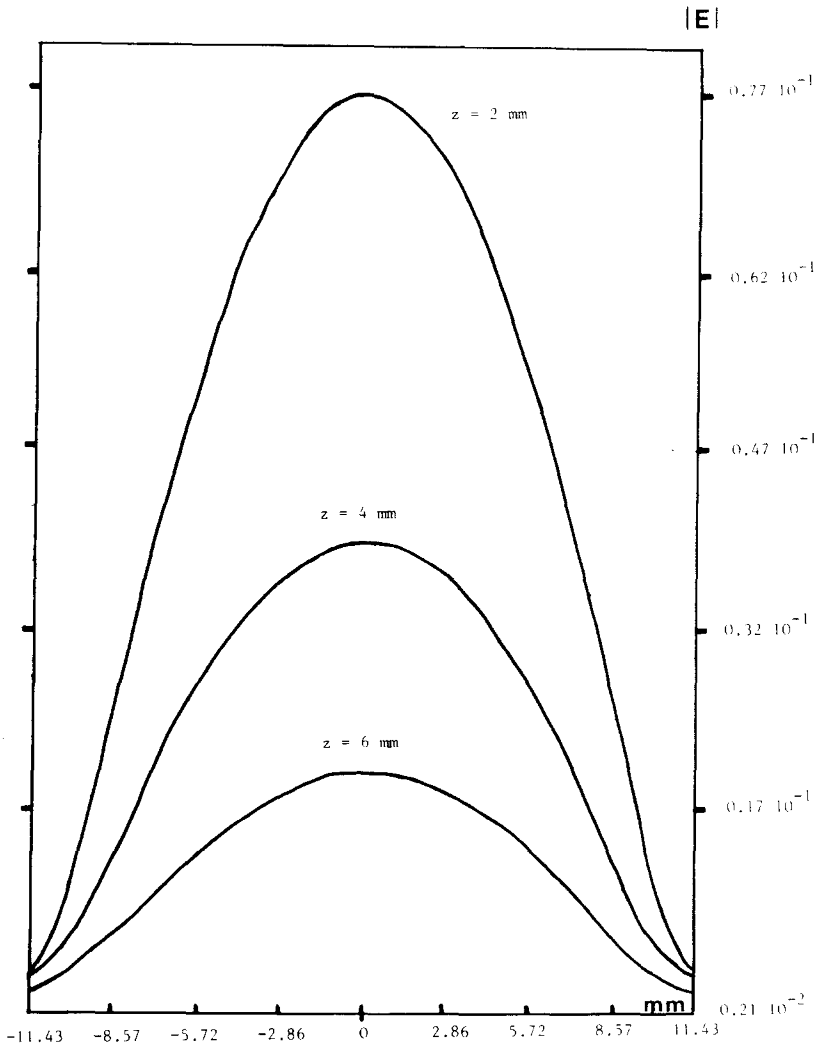


FIG. 2. Modulus of the diffracted field at different distances from the aperture for the $\varepsilon_1 = \varepsilon_0$, $\sigma_1 = 0$, $\varepsilon_2 = 70\varepsilon_0$, and $\sigma_2 = 15$ guide.

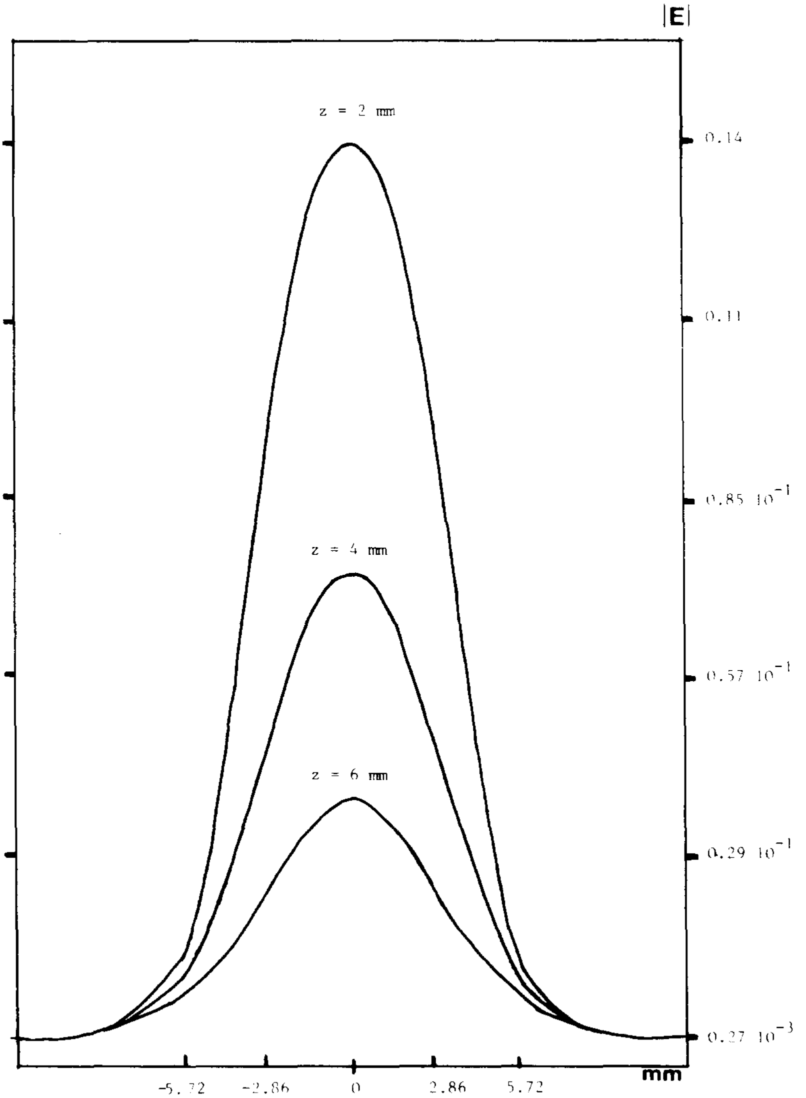


FIG. 3. Modulus of the diffracted field at different distances from the aperture for the $\varepsilon_1 = 4\varepsilon_0$, $\sigma_1 = 0$, $\varepsilon_2 = 70\varepsilon_0$, and $\sigma_2 = 15$ guide.

integral equation with Cauchy principal values and the solution of the flanged waveguide problem. This method has been found to be table, accurate and convergent.

Singular integral equations arise frequently in electromagnetic and acoustic theory and the methods described here could be applied to such problems as radiation from antennas with circular symmetry, radiation by a slot or a screen, and scattering by cilindrical bodies, for example.

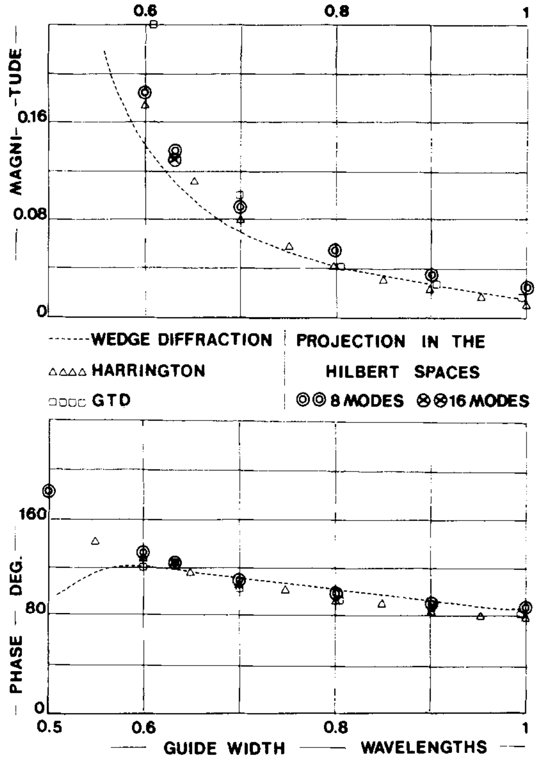


FIG. 4. Comparison between wedge diffraction [11], Harrington [22], geometrical theory of diffraction (GTD[23]) and projection in Hilbert spaces for the amplitude B_1 of the first reflected mode.

APPENDIX: THE RADIATION CHARACTERISTIC AND ENERGY CONSERVATION

In Fig. 1, the field in region $z > 0$ is obtained by replacing $u_2(x', 0)$ with its modal expression in expression(5):

$$u_2(x, z) = 2 \sum_m (A_m + B_m) \int_{-a/2}^{a/2} \phi_m(x') \, \partial_{z'} G(x, x'; z, z')|_{z'=0} \, dx',$$

where

$$\partial_{z'} G(x, x'; z, z')|_{z'=0} = \frac{ik_2}{2} \frac{z}{\sqrt{(x-x')^2 + z^2}} H_1^{(1)}(k_2 \sqrt{(x-x') + z^2}).$$

When the observation point (x, z) is sufficiently distant, we can use an asymptotic form for the Hankel function. Then we have, in polar coordinates,

$$u(r, \phi) = \sqrt{\frac{2}{\pi k_2 r}} \, e^{i(k_2 r - 3\pi/4)} \hat{u}_0(\phi),$$

where

$$\hat{u}_0(\phi) = ik_2/2 \sum_m (A_m + B_m) \cos \phi J_m(\phi)$$

is the radiation characteristic of the waveguide, in which

$$J_{2K}(\phi) = i(-1)^K 4\pi K a \frac{\sin(k_2 a/2 \sin \phi)}{(k_2 a \sin \phi)^2 - (2K\pi)^2},$$

$$J_{2K+1}(\phi) = (-1)^{K+1} 2(2K+1)\pi a \frac{\cos(k_2 a/2 \sin \phi)}{(k_2 a \sin \phi) - (2K+1)^2 \pi^2}.$$

The power radiated in a non-dissipative medium is obtained by integration of the complex Poynting vector over a closed surface. The following expression results,

$$\sum_m (|A_m|^2 - |B_m|^2) \operatorname{Im}(\gamma_m) + \frac{4\mu_1}{\mu_2} \int_{-\pi/2}^{\pi/2} |\hat{u}_0(\phi)|^2 \, d\phi = 0,$$

where $\operatorname{Im}(\gamma_m)$ is the imaginary part of γ_m. In the particular case

$$A_1 = 1 \quad \text{and} \quad A_m = 0 \quad \text{for} \quad m \neq 1$$

we have

$$\frac{1}{\operatorname{Im}(\gamma_1)} \left\{ \sum_m |B_m|^2 \operatorname{Im}(\gamma_m) \right\} - \frac{4\mu_1}{\operatorname{Im}(\gamma_1)\pi a \mu_2} \int_{-\pi/2}^{\pi/2} |\hat{u}_0(\phi)|^2 \, d\phi = 1.$$

ACKNOWLEDGMENTS

The authors wish to thank Professor Cherruault, Professor Roubine and Professor Bolomey for their guidance and encouragement and Professor Bouix of the Centre d'Etudes Théoriques de la Détection et des Communications, which has sponsored this work.

REFERENCES

1. L. SCHWARTZ, "Théorie des distributions," Hermann, Paris, 1966.
2. J. VAN BLADEL, *IRE Trans. A-P* (1961), 563–566.
3. J. G. FIKIORIS, *J. Math. Phys.* **6** No. 11 (1965), 1617–1620.
4. P. FILIPPI AND G. DUMERY, *Acustica* **21** (1969), 344–350.
5. V. S. VLADIMIROV, "Equations of Mathematical Physics," Dekker, New York, 1971.
6. B. COLOMBEAU, P. FACQ, AND J. DU MARACHE, *Optica Acta* **20**, No. 12 (1973), 917–924.
7. F. M. TESCHE, *IEEE Trans. Elec. Comp.* **16** (1974), 203.
8. M. VAMPOUILLE AND P. FACQ, *Optica Acta* **21** No. 7 (1974), 595.
9. A. J. POGGIO, Report UCRL 51722, Lawrence Livermore Laboratory, 1975.
10. W. A. DAVIS AND R. MITTRA, *IEEE Trans. Antennas and Propagation* **AP-25** (1977), 402–406.
11. R. C. RUDDUCK AND L. L. TSAI, *IEEE Trans. Antennas and Propagation* **AP-16**, No. 1 (1968).

12. C. P. Wu, *IEEE Trans. Antennas and Propagation* **AP–17**, No. 6 (1969), 733–739.
13. J. J. Bowman, *IEEE Trans. Antennas and Propagation* **AP–18**, No. 1 (1970), 131–132.
14. T. Itoh and R. Mittra, *Proc. IEEE* **57**, No. 7 (1971), 1131–1133.
15. E. V. Jull, *IEEE Trans. Antennas and Propagation* **AP–21**, No. (1973), 14–18.
16. J. Ch. Bolomey, Scientific Report 1974, Lab. d'Electromagnétisme, LSS–ESE, Plateau du Moulon, Gif-sur-Yvette 91190, France.
17. L. E. Rikenglaz and O. V. Kolokolova, *Rad. Eng. Elect. Phys.* **20** (1975), 125–128.
18. P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," MacGraw–Hill New York, 1953.
19. M. Abramovitz and I. A. Stegun, "Handbook of Mathematical Functions," Dover, New York, 1970.
20. Y. Cherruault, "Approximations d'opérateurs linéaires et applications," Dunod, Paris, 1968.
21. G. Nissen, Thèse de Doctorat, Paris, 1968.
22. R. F. Harrington, "Time-Harmonic Electromagnetic Fields," pp. 180–185, Mac Graw–Hill, New York, 1961.
23. H. Y. Yee, L. B. Felsen and J.B. Keller, *SIAM J. Appl. Math.* **16** (1968), 268–300.